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Generalized Inverses and Ranks of Modified Matrices[1]

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SUMMARY

The problem considered is that of obtaining generalized inverses and ranks of matrices of the form $\mathbf{R}+\mathbf{STU}$ (in terms of generalized inverses of $\mathbf{R}$ and various other matrices or in terms of the ranks of these matrices). It is found that generalized inverses of $\mathbf{R}+\mathbf{STU}$ can be obtained as submatrices of generalized inverses of the partitioned matrix

$$\mathbf{C}=\begin{pmatrix}\mathbf{R} & -\mathbf{ST}\\ \mathbf{TU} & \mathbf{T}\end{pmatrix}$$

and that $\text{rank}(\mathbf{R}+\mathbf{STU})=\text{rank}(\mathbf{C})-\text{rank}(\mathbf{T})$. These results are used to translate various formulas for generalized inverses and ranks of partitioned matrices into formulas for generalized inverses of $\mathbf{R}+\mathbf{STU}$ and for $\text{rank}(\mathbf{R}+\mathbf{STU})$. The formulas for generalized inverses of $\mathbf{R}+\mathbf{STU}$ can be regarded as generalizations of Woodbury's formula (which is for an ordinary inverse).

Key words : Woodbury's formula, Matrix sum, Partitioned matrix, Recursive estimation.

1. *Introduction*

Let $\mathbf{R}$ represent an $n\times q$ matrix, S an $n\times m$ matrix, $\mathbf{T}$ an $m\times p$ matrix, and $\mathbf{U}$ a $p\times q$ matrix, and consider the modified matrix $\mathbf{R}+\mathbf{STU}$ obtained by adding to $\mathbf{R}$ the product $\mathbf{STU}$.

In the special case where $\mathbf{R}$ and $\mathbf{T}$ are nonsingular, it is well-known that $\mathbf{R}+\mathbf{STU}$ is nonsingular if and only if $\mathbf{T}^{-1}+\mathbf{UR}^{-1}\mathbf{S}$ is nonsingular, or equivalently if and only if $\mathbf{T}+\mathbf{TUR}^{-1}\mathbf{ST}$ is nonsingular, in which case

$$(\mathbf{R}+\mathbf{STU})^{-1}=\mathbf{R}^{-1}-\mathbf{R}^{-1}\mathbf{S}(\mathbf{T}^{-1}+\mathbf{UR}^{-1}\mathbf{S})^{-1}\mathbf{UR}^{-1} \quad (1.1)$$

$$=\mathbf{R}^{-1}-\mathbf{R}^{-1}\mathbf{ST}(\mathbf{T}+\mathbf{TUR}^{-1}\mathbf{ST})^{-1}\mathbf{TUR}^{-1} \quad (1.2)$$

Formula (1.1) can be useful in instances where $\mathbf{R}$ is easy to invert (as would be the case if, e.g., $\mathbf{R}$ were diagonal) or has already been inverted and where

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the dimensions of **T** (and hence those of $\mathbf{T}^{-1} + \mathbf{U}\mathbf{R}^{-1}\mathbf{S}$) are small relative to those of **R**. Formula (1.2) calls for more matrix multiplications than formula (1.1), but does not call for the inversion of **T**.

As discussed by Henderson and Searle [7], formulae (1.1) and (1.2) have many applications in statistics. Formulae (1.1) and (1.2) were given by Woodbury [16], and one or the other of them is often referred to as Woodbury's formula. The special case of formula (1.1) or (1.2) where **T** is the 1×1 matrix (1) is often attributed to Sherman and Morrison [14], [15] and/or to Bartlett [2]. Refer to Henderson and Searle or to Ouellette (1981, Sec. 2.3) for additional information about the history of formulae (1.1) and (1.2).

In some applications, **R** and/or **T** may not be nonsingular (and may not even be square), in which case there may be a need for formulae for a generalized inverse of $\mathbf{R} + \mathbf{STU}$ [that are comparable to formulae (1.1) and (1.2) for an ordinary inverse], and there may also be a need for a formula that relates rank($\mathbf{R} + \mathbf{STU}$) to rank(**R**). Here, generalized inverse means "weak" generalized inverse, that is, a generally nonunique matrix that satisfies the first of the Penrose conditions. And, for any matrix **A**, the symbol $\mathbf{A}^-$ is used to denote an arbitrary generalized inverse of **A**, that is, an arbitrary solution to $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$. Typically, a weak generalized inverse of $\mathbf{R} + \mathbf{STU}$ is sufficient for statistical applications.

Consider, for example, a statistical application in which data are acquired and analyzed in two stages. Let $\mathbf{y}_1$ represent an $n_1 \times 1$ vector whose elements are the (real-valued) data points from that first stage, and $\mathbf{y}_2$ an $n_2 \times 1$ vector whose elements are the data points from the second stage. Suppose that $\mathbf{y}_i$ is regarded as a random vector that follows the linear statistical model $\mathbf{y}_i = \mathbf{X}_i\beta + \mathbf{e}_i$ (i = 1, 2), where $\mathbf{X}_i$ is an $n_i \times k$ (known) matrix of rank r_i, β is a vector of unknown parameters, $\mathbf{e}_i$ is an unobservable random vector with mean **0** and variance-covariance matrix $\sigma^2\mathbf{H}_i$, σ^2 is a known or unknown positive scalar, and $\mathbf{H}_i$ is a (known) positive definite matrix. Further, suppose that $\mathbf{e}_1$ and $\mathbf{e}_2$ are uncorrelated; let

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} \mathbf{H}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_2 \end{pmatrix}$$

and r = rank(**X**); and take Λ_1 and Λ to be any matrices (with k rows) such that $\Re(\Lambda'_1) \subset \Re(\mathbf{X}_1)$ and $\Re(\Lambda') \subset \Re(\mathbf{X})$. [For any matrix **A**, the row and column spaces of **A** are denoted herein by $\Re(\mathbf{A})$ and $\mathcal{C}(\mathbf{A})$, respectively.]

During the initial (first-stage) analysis, $\Lambda'_1 \beta$ can be estimated unbiasedly (from y_1) by the generalized least squares estimator $\Lambda'_1 (X'_1 H_1^{-1} X_1)^- X'_1 H_1^{-1} y_1$; and during the final (second-stage) analysis, $\Lambda' \beta$ can be estimated unbiasedly (from y) by the generalized least squares estimator $\Lambda' (X' H^{-1} X)^- X' H^{-1} y$. And, the variance-covariance matrices of these two estimators are $\sigma^2 \Lambda'_1 (X'_1 H_1^{-1} X_1)^- \Lambda_1$ and $\sigma^2 \Lambda' (X' H^{-1} X)^- \Lambda$, respectively.

Now, set $R = X'_1 H_1^{-1} X_1$, $S = X'_2$, $T = H_2^{-1}$, and $U = X_2$. Then, $R + STU = X' H^{-1} X$ [and $\text{rank}(R) = r_1$ and $\text{rank}(R + STU) = r$], and an expression for a generalized inverse of $X'H^{-1}X$ in terms of a generalized inverse of **R** (and an expression for r in terms of r_1) may be of interest. In particular, if n_2 is small relative to k, then (since a generalized inverse of **R** may be available from the first-stage analysis) such an expression may be useful for computational purposes.

The objective in the present paper is to extend Woodbury's formula to generalized inverses and to obtain some related results on ranks. Previously, Meyer ([9], sec. 4), proceeding on a case-by-case basis, obtained an expression for a generalized inverse of $R + STU$ (in terms of a generalized inverse of **R**) in the special case where $m = p = 1$ (i.e., where **S**, **T**, and **U** are respectively of dimensions $n \times 1$, 1×1, and $1 \times q$) and where $T = (1)$. And, Rao and Mitra ([13], pp. 70-71) indicated that, in the event that $\mathfrak{R}(U) \subset \mathfrak{R}(R)$ or $\mathcal{C}(S) \subset \mathcal{C}(R)$ (and **T** and $T^{-1} + UR^-S$ are nonsingular), formula (1.1) can be generalized simply by substituting R^- for R^{-1}. Further, Henderson and Searle ([7], sec. 4), following Harville [6], listed some formulae for generalized inverses (of $R + STU$) that are applicable whenever $\mathfrak{R}(STU) \subset \mathfrak{R}(R)$ and $\mathcal{C}(STU) \subset \mathcal{C}(R)$.

Section 2 of the present paper provides a systematic basis for extending Woodbury's formula and for obtaining related results on ranks. It does so by establishing that generalized inverses of $R + STU$ can be obtained as submatrices of the generalized inverses of a certain partitioned matrix and by relating rank $(R + STU)$ to the rank of that partitioned matrix. In Sections 3 and 4, the results of Section 2 are used in combination with various results on partitioned matrices to devise formulae for generalized inverses of $R + STU$ and for rank $(R + STU)$. The formulae given in Section 3 are relatively simple, but are only applicable in special cases; those given in Section 4 are more complex, but apply without restriction.

2. *Some Connections between Generalized Inverses and Ranks of Modified Matrices and those of Partitioned Matrices*

Preliminary to establishing (as Theorem 2.5) the main result of this section, it is convenient to state (in the form of the following theorem) some basic results on generalized inverses and ranks of Schur complements (in partitioned matrices).

Theorem 2.1. Let $\mathbf{E}$ represent an $r \times t$ matrix, $\mathbf{F}$ an $r \times u$ matrix, $\mathbf{V}$ an $s \times t$ matrix, and $\mathbf{W}$ an $s \times u$ matrix, and define $\mathbf{Q} = \mathbf{W} - \mathbf{V}\mathbf{E}^{-}\mathbf{F}$. Suppose that $\Re(\mathbf{V}) \subset \Re(\mathbf{E})$ and $\mathcal{C}(\mathbf{F}) \subset \mathcal{C}(\mathbf{E})$. Then, for any generalized inverse $\mathbf{G} = \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{pmatrix}$ of the partitioned matrix $\begin{pmatrix} \mathbf{W} & \mathbf{V} \\ \mathbf{F} & \mathbf{E} \end{pmatrix}$, the $(u \times s)$ submatrix $\mathbf{G}_{11}$ is a generalized inverse of $\mathbf{Q}$. Similarly, for any generalized inverse $\mathbf{H} = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{pmatrix}$ of the partitioned matrix $\begin{pmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{V} & \mathbf{W} \end{pmatrix}$, the $(u \times s)$ submatrix $\mathbf{H}_{22}$ is a generalized inverse of $\mathbf{Q}$. Further,

$$\text{rank}\begin{pmatrix} \mathbf{W} & \mathbf{V} \\ \mathbf{F} & \mathbf{E} \end{pmatrix} = \text{rank}(\mathbf{E}) + \text{rank}(\mathbf{Q}) \tag{2.1}$$

$$\text{rank}\begin{pmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{V} & \mathbf{W} \end{pmatrix} = \text{rank}(\mathbf{E}) + \text{rank}(\mathbf{Q}) \tag{2.2}$$

The results of Theorem 2.1 are available in the literature. For results (2.1) and (2.2), refer, for instance, to Marsaglia and Styan ([8], Cor. 19.1) or Carlson ([4], pp. 262). With regard to the other parts of Theorem 2.1 (the parts pertaining to generalized inverses), it follows from the results of Bhimasankaram ([3], Theorem 2) that there exists a generalized inverse of $\begin{pmatrix} \mathbf{W} & \mathbf{V} \\ \mathbf{F} & \mathbf{E} \end{pmatrix}$ having a generalized inverse of $\mathbf{Q}$ in its upper left corner and similarly that there exists a generalized inverse of $\begin{pmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{V} & \mathbf{W} \end{pmatrix}$ having a generalized inverse of $\mathbf{Q}$ in its lower right corner. That every generalized inverse of $\begin{pmatrix} \mathbf{W} & \mathbf{V} \\ \mathbf{F} & \mathbf{E} \end{pmatrix}$ has a generalized inverse of $\mathbf{Q}$ in its upper left corner and every generalized inverse of $\begin{pmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{V} & \mathbf{W} \end{pmatrix}$ has a generalized inverse of $\mathbf{Q}$ in its lower right corner (as claimed by Theorem 2.1) was established by Mitra ([11], Lemma 6) — refer also to Balasubramanian, Dey, and Bhimasankaram ([1], Lemma 7). A proof of Theorem 2.1 is included for completeness. This proof makes use of the following three well-known (and easily verifiable) lemmas.

Lemma 2.2. Let $\mathbf{A}$ represent an $r \times s$ matrix. Then, for any $r \times t$ matrix $\mathbf{B}$, $\mathcal{C}(\mathbf{B}) \subset \mathcal{C}(\mathbf{A})$ if and only if $\mathbf{B} = \mathbf{A}\mathbf{A}^{-}\mathbf{B}$. And, for any $u \times s$ matrix $\mathbf{C}$, $\Re(\mathbf{C}) \subset \Re(\mathbf{A})$ if and only if $\mathbf{C} = \mathbf{C}\mathbf{A}^{-}\mathbf{A}$.

Lemma 2.3. Let $\mathbf{B}$ represent an $r \times s$ matrix, and $\mathbf{G}$ an $s \times r$ matrix. Then, for any $r \times r$ nonsingular matrix $\mathbf{A}$ and $s \times s$ nonsingular matrix $\mathbf{C}$, $\mathbf{G}$ is a generalized inverse of $\mathbf{ABC}$ if and only if $\mathbf{G} = \mathbf{C}^{-1}\mathbf{H}\mathbf{A}^{-1}$ for some generalized inverse $\mathbf{H}$ of $\mathbf{B}$.

Lemma 2.4. Let $\mathbf{E}$ represent an $r \times t$ matrix, and $\mathbf{W}$ an $s \times u$ matrix. Then, a partitioned matrix $\mathbf{G} = \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{pmatrix}$ (where $\mathbf{G}_{11}$ is of dimension $u \times s$) is a generalized inverse of the $(s+r) \times (u+t)$ block-diagonal partitioned matrix $\begin{pmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{pmatrix}$ if and only if $\mathbf{W}\mathbf{G}_{11}\mathbf{W} = \mathbf{W}$ (i.e., $\mathbf{G}_{11}$ is a generalized inverse of $\mathbf{W}$), $\mathbf{E}\mathbf{G}_{22}\mathbf{E} = \mathbf{E}$ (i.e., $\mathbf{G}_{22}$ is a generalized inverse of $\mathbf{E}$), $\mathbf{W}\mathbf{G}_{12}\mathbf{E} = \mathbf{0}$, and $\mathbf{E}\mathbf{G}_{21}\mathbf{W} = \mathbf{0}$.

Proof (of Theorem 2.1). Observing (in light of Lemma 2.2) that $\mathbf{V} - \mathbf{V}\mathbf{E}^{-}\mathbf{E} = \mathbf{0}$, and that $\mathbf{F} - \mathbf{E}\mathbf{E}^{-}\mathbf{F} = \mathbf{0}$, we find that

$$\begin{pmatrix} \mathbf{I} & -\mathbf{V}\mathbf{E}^{-} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{W} & \mathbf{V} \\ \mathbf{F} & \mathbf{E} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{E}^{-}\mathbf{F} & \mathbf{I} \end{pmatrix} = \begin{pmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{pmatrix}$$

Since the pre- or post-multiplication of a matrix by a nonsingular matrix does not affect its rank, we have that

$$\text{rank}\begin{pmatrix} \mathbf{W} & \mathbf{V} \\ \mathbf{F} & \mathbf{E} \end{pmatrix} = \text{rank}\begin{pmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{pmatrix} = \text{rank}(\mathbf{E}) + \text{rank}(\mathbf{Q})$$

Further, it follows from Lemma 2.3 that the matrix

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{E}^{-}\mathbf{F} & \mathbf{I} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\mathbf{V}\mathbf{E}^{-} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{E}^{-}\mathbf{F} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{V}\mathbf{E}^{-} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} + \mathbf{G}_{11}\mathbf{V}\mathbf{E}^{-} \\ \mathbf{G}_{21} + \mathbf{E}^{-}\mathbf{F}\mathbf{G}_{11} & \mathbf{G}_{22} + \mathbf{E}^{-} + \mathbf{F}\mathbf{G}_{12} + \mathbf{G}_{21}\mathbf{V}\mathbf{E}^{-} + \mathbf{E}^{-}\mathbf{F}\mathbf{G}_{11}\mathbf{V}\mathbf{E}^{-} \end{pmatrix}$$

is a generalized inverse of the matrix $\begin{pmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{pmatrix}$, implying (in light of Lemma 2.4) that $\mathbf{G}_{11}$ is a generalized inverse of $\mathbf{Q}$. The validity of result (2.2) and of the claim that $\mathbf{H}_{22}$ is a generalized inverse of $\mathbf{Q}$ can be established in similar fashion.

Let

$$\mathbf{C} = \begin{pmatrix} \mathbf{R} & -\mathbf{S}\mathbf{T} \\ \mathbf{T}\mathbf{U} & \mathbf{T} \end{pmatrix}$$

Upon applying Theorem 2.1 (with $\mathbf{E} = \mathbf{T}$, $\mathbf{F} = \mathbf{T}\mathbf{U}$, $\mathbf{V} = -\mathbf{S}\mathbf{T}$, and $\mathbf{W} = \mathbf{R}$), we obtain the following very useful result.

Theorem 2.5. For any generalized inverse $\mathbf{G} = \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{pmatrix}$ of the matrix $\mathbf{C}$, the $(q \times n)$ submatrix $\mathbf{G}_{11}$ is a generalized inverse of $\mathbf{R} + \mathbf{S}\mathbf{T}\mathbf{U}$. Further,

$$\text{rank}\,(\mathbf{R} + \mathbf{S}\mathbf{T}\mathbf{U}) = \text{rank}\,(\mathbf{C}) - \text{rank}\,(\mathbf{T}) \qquad (2.3)$$

As demonstrated in Sections 3 and 4, Theorem 2.5 provides a very convenient and very effective vehicle for translating formulae for generalized inverses and ranks of partitioned matrices into formulae for generalized inverses of $\mathbf{R} + \mathbf{S}\mathbf{T}\mathbf{U}$ and for rank $(\mathbf{R} + \mathbf{S}\mathbf{T}\mathbf{U})$.

3. *Some Special Cases*

Let

$$\mathbf{Q} = \mathbf{T} + \mathbf{T}\mathbf{U}\mathbf{R}^{-}\mathbf{S}\mathbf{T}$$

and consider the matrix

$$\mathbf{R}^- - \mathbf{R}^- \mathbf{S}\mathbf{T}\mathbf{Q}^- \mathbf{T}\mathbf{U}\mathbf{R}^- \tag{3.1}$$

which is the matrix obtained from formula (1.2) by replacing ordinary inverses with generalized inverses. Under certain conditions, the matrix $\mathbf{C}$ (introduced in Section 2) has a generalized inverse $\mathbf{G} = \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{pmatrix}$ for which $\mathbf{G}_{11}$ equals matrix (3.1). For $\mathbf{C}$ to have such a generalized inverse, it is sufficient that $\mathcal{R}(\mathbf{TU}) \subset \mathcal{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{ST}) \subset \mathcal{C}(\mathbf{R})$ (e.g. Bhimasankaram [3], Theorem 2). Thus, it follows from Theorem 2.5 that if $\mathfrak{R}(\mathbf{TU}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{ST}) \subset \mathcal{C}(\mathbf{R})$, matrix (3.1) is a generalized inverse of $\mathbf{R}+\mathbf{STU}$.

While the conditions $\mathfrak{R}(\mathbf{TU}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{ST}) \subset \mathcal{C}(\mathbf{R})$ are sufficient for matrix (3.1) to be a generalized inverse of $\mathbf{R}+\mathbf{STU}$, they are not necessary. After some algebraic manipulation, we find that

$$\begin{aligned}(\mathbf{R}+\mathbf{STU})&(\mathbf{R}^- - \mathbf{R}^-\mathbf{STQ}^-\mathbf{TUR}^-)(\mathbf{R}+\mathbf{STU})\\ &= \mathbf{R}+\mathbf{STU} - \mathbf{S}(\mathbf{I}-\mathbf{QQ}^-)\mathbf{TU}(\mathbf{I}-\mathbf{R}^-\mathbf{R})\\ &\quad - (\mathbf{I}-\mathbf{RR}^-)\mathbf{ST}(\mathbf{I}-\mathbf{Q}^-\mathbf{Q})\mathbf{U} - (\mathbf{I}-\mathbf{RR}^-)\mathbf{STQ}^-\mathbf{TU}(\mathbf{I}-\mathbf{R}^-\mathbf{R})\end{aligned}$$

Thus, we have the following theorem.

Theorem 3.1. For matrix (3.1) to be a generalized inverse of $\mathbf{R}+\mathbf{STU}$, it is necessary and sufficient that

$$\begin{aligned}\mathbf{S}(\mathbf{I}-\mathbf{QQ}^-)\mathbf{TU}(\mathbf{I}-\mathbf{R}^-\mathbf{R}) &+ (\mathbf{I}-\mathbf{RR}^-)\mathbf{ST}(\mathbf{I}-\mathbf{Q}^-\mathbf{Q})\mathbf{U}\\ &+ (\mathbf{I}-\mathbf{RR}^-)\mathbf{STQ}^-\mathbf{TU}(\mathbf{I}-\mathbf{R}^-\mathbf{R}) = \mathbf{0}\end{aligned} \tag{3.2}$$

Condition (3.2) is of course satisfied if $\mathfrak{R}(\mathbf{TU}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{ST}) \subset \mathcal{C}(\mathbf{R})$. In fact, it is satisfied if $\mathfrak{R}(\mathbf{STU}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{STU}) \subset \mathcal{C}(\mathbf{R})$. To see this, consider the following lemma.

Lemma 3.2. If $\mathfrak{R}(\mathbf{STU}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{STU}) \subset \mathcal{C}(\mathbf{R})$, then

$$(\mathbf{I}-\mathbf{QQ}^-)\mathbf{TU}(\mathbf{I}-\mathbf{R}^-\mathbf{R}) = \mathbf{0} \tag{3.3}$$

$$(\mathbf{I}-\mathbf{RR}^-)\mathbf{ST}(\mathbf{I}-\mathbf{Q}^-\mathbf{Q}) = \mathbf{0} \tag{3.4}$$

$$(\mathbf{I}-\mathbf{RR}^-)\mathbf{STQ}^-\mathbf{TU}(\mathbf{I}-\mathbf{R}^-\mathbf{R}) = \mathbf{0} \tag{3.5}$$

Proof. Let $\mathbf{D}=\mathbf{STU}$, and observe that $\mathbf{QQ}^-(\mathbf{T}+\mathbf{TUR}^-\mathbf{ST}) = \mathbf{T}+\mathbf{TUR}^-\mathbf{ST}$, or equivalently that $(\mathbf{I}-\mathbf{QQ}^-)\mathbf{T} = -(\mathbf{I}-\mathbf{QQ}^-)\mathbf{TUR}^-\mathbf{ST}$, and hence that

$$(\mathbf{I} - \mathbf{QQ}^-)\mathbf{TU} = -(\mathbf{I} - \mathbf{QQ}^-)\mathbf{TUR}^-\mathbf{D} \tag{3.6}$$

It can be shown in similar fashion that

$$\mathbf{ST}(\mathbf{I} - \mathbf{Q}^-\mathbf{Q}) = -\mathbf{DR}^-\mathbf{ST}(\mathbf{I} - \mathbf{Q}^-\mathbf{Q}) \tag{3.7}$$

Further, $(\mathbf{T} + \mathbf{TUR}^-\mathbf{ST})\mathbf{Q}^-(\mathbf{T} + \mathbf{TUR}^-\mathbf{ST}) = \mathbf{T} + \mathbf{TUR}^-\mathbf{ST}$, or equivalently

$$\mathbf{TQ}^-\mathbf{T} = \mathbf{T} + \mathbf{TUR}^-\mathbf{ST} - \mathbf{TUR}^-\mathbf{STQ}^-\mathbf{T} - \mathbf{TQ}^-\mathbf{TUR}^-\mathbf{ST}$$
$$- \mathbf{TUR}^-\mathbf{STQ}^-\mathbf{TUR}^-\mathbf{ST}$$

implying that

$$\mathbf{STQ}^-\mathbf{TU} = \mathbf{D} + \mathbf{DR}^-\mathbf{D} - \mathbf{DR}^-\mathbf{STQ}^-\mathbf{TU}$$
$$- \mathbf{STQ}^-\mathbf{TUR}^-\mathbf{D} - \mathbf{DR}^-\mathbf{STQ}^-\mathbf{TUR}^-\mathbf{D} \tag{3.8}$$

And, if $\mathfrak{R}(\mathbf{D}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{D}) \subset \mathcal{C}(\mathbf{R})$ [in which case $\mathbf{D}(\mathbf{I} - \mathbf{R}^-\mathbf{R}) = \mathbf{0}$ and $(\mathbf{I} - \mathbf{RR}^-)\mathbf{D} = \mathbf{0}$], results (3.3)-(3.5) follow from results (3.6)-(3.8).

If $\mathfrak{R}(\mathbf{STU}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{STU}) \subset \mathcal{C}(\mathbf{R})$, then it follows from Lemma 3.2 that condition (3.2) is satisfied. Thus, as a corollary of Theorem 3.1, we have the following result, noted previously by Henderson and Searle ([7], sec. 4).

Corollary 3.3. For matrix (3.1) to be a generalized inverse of $\mathbf{R} + \mathbf{STU}$, it is sufficient that $\mathfrak{R}(\mathbf{STU}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{STU}) \subset \mathcal{C}(\mathbf{R})$.

There is an alternative way to use Lemma 3.2 to establish Corollary 3.3. It follows from Theorem 1 of Bhimasankaram [3] that, for the matrix $\mathbf{C}$ to have a generalized inverse with matrix (3.1) in the upper left corner, it is sufficient that conditions (3.3)—(3.5) be satisfied. Thus, if $\mathfrak{R}(\mathbf{STU}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{STU}) \subset \mathcal{C}(\mathbf{R})$, then (in light of Lemma 3.2) $\mathbf{C}$ has such a generalized inverse, implying (in light of Theorem 2.5) that matrix (3.1) is a generalized inverse of $\mathbf{R} + \mathbf{STU}$.

Under the conditions of Corollary 3.3, variations on formula (3.1) can be obtained by rewriting $\mathbf{R} + \mathbf{STU}$ and $\mathbf{R} + (\mathbf{ST})\mathbf{T}^-(\mathbf{TU}), \mathbf{R} + (\mathbf{ST})\mathbf{I}_p\mathbf{U}$, or $\mathbf{R} + \mathbf{SI}_m(\mathbf{TU})$ and by applying formula (3.1) with $\mathbf{ST}, \mathbf{T}^-$, and $\mathbf{TU}$; $\mathbf{ST}, \mathbf{I}_p$, and $\mathbf{U}$; or $\mathbf{S}, \mathbf{I}_m$, and $\mathbf{TU}$ in place of $\mathbf{S}, \mathbf{T}$, and $\mathbf{U}$. We find that, in the special case where $\mathfrak{R}(\mathbf{STU}) \subset \mathfrak{R}(\mathbf{T})$ and $\mathcal{C}(\mathbf{STU}) \subset \mathcal{C}(\mathbf{R})$, each of the following matrices is a generalized inverse of $\mathbf{R} + \mathbf{STU}$:

$$\mathbf{R}^- - \mathbf{R}^-\mathbf{STT}^-(\mathbf{T}^- + \mathbf{T}^-\mathbf{TUR}^-\mathbf{STT}^-)^-\mathbf{T}^-\mathbf{TUR}^-$$
$$\mathbf{R}^- - \mathbf{R}^-\mathbf{ST}(\mathbf{I}_p + \mathbf{UR}^-\mathbf{ST})^-\mathbf{UR}^-$$
$$\mathbf{R}^- - \mathbf{R}^-\mathbf{S}(\mathbf{I}_m + \mathbf{TUR}^-\mathbf{S})^-\mathbf{TUR}^- \tag{3.9}$$

Note that, in the special case where $\mathbf{R}, \mathbf{T}$, and $\mathbf{T}+\mathbf{TUR}^{-}\mathbf{ST}$ are nonsingular, formula (3.9) reduces to formula (1.1).

Now, consider the rank of $\mathbf{R}+\mathbf{STU}$. When $\mathfrak{R}(\mathbf{TU}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{ST}) \subset \mathcal{C}(\mathbf{R})$, it follows from result (2.2) that

$$\text{rank}(\mathbf{C}) = \text{rank}(\mathbf{R}) + \text{rank}(\mathbf{Q}) \qquad (3.10)$$

More generally, in light of Lemma 3.2, it follows from Corollary 19.1 of Marsaglia and Styan [8] that equality (3.10) holds when $\mathfrak{R}(\mathbf{STU}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{STU}) \subset \mathcal{C}(\mathbf{R})$. Thus, as a consequence of Theorem 2.5, we have the following result.

Theorem 3.4. If $\mathfrak{R}(\mathbf{TU}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{ST}) \subset \mathcal{C}(\mathbf{R})$ or more generally if $\mathfrak{R}(\mathbf{STU}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathrm{C}(\mathbf{STU}) \subset \mathrm{C}(\mathbf{R})$, then

$$\text{rank}(\mathbf{R}+\mathbf{STU}) = \text{rank}(\mathbf{R}) - [\text{rank}(\mathbf{T}) - \text{rank}(\mathbf{Q})] \qquad (3.11)$$

In connection with Theorem 3.4, note that $\mathbf{Q} = \mathbf{T}(\mathbf{T}^{-}\mathbf{T}+\mathbf{UR}^{-}\mathbf{ST})$ and hence that (regardless of whether the conditions of the theorem are satisfied) $\text{rank}(\mathbf{T}) - \text{rank}(\mathbf{Q}) \geq 0$. Note also that equality (3.11) is equivalent to the equality

$$\text{rank}(\mathbf{R}) - \text{rank}(\mathbf{R}+\mathbf{STU}) = \text{rank}(\mathbf{T}) - \text{rank}(\mathbf{Q})$$

Thus, the difference in rank between $\mathbf{T}$ and $\mathbf{Q}$ is non-negative and, when $\mathfrak{R}(\mathbf{STU}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{STU}) \subset \mathcal{C}(\mathbf{R})$, is the same as the difference in rank between $\mathbf{R}$ and $\mathbf{R}+\mathbf{STU}$.

4. *General Case : Some New Formulae*

In Section 3, it was found that, in the special case where $\mathfrak{R}(\mathbf{STU}) \subset \mathfrak{R}(\mathbf{R})$ and $\mathcal{C}(\mathbf{STU}) \subset \mathcal{C}(\mathbf{R})$, condition (3.2) [which is necessary and sufficient for formula (3.1) to be applicable] is satisfied (without any restriction on the choice of the generalized inverses $\mathbf{R}^{-}$ and $\mathbf{Q}^{-}$). However, aside from that very important special case, the restrictions imposed by condition (3.2) would appear to be rather severe. Consider, for example, the special case where $m = p = 1$ (so that $\mathbf{S}$ is a column vector and $\mathbf{U}$ a row vector and $\mathbf{T}$ and $\mathbf{Q}$ are scalars) and where $\mathbf{T} = (1)$, $\mathbf{S} \notin \mathcal{C}(\mathbf{T})$, and $\mathbf{U} \notin \mathfrak{R}(\mathbf{T})$. In that special case, condition (3.2) implies that $\mathbf{Q} = (0)$ and $\mathbf{Q}^{-} = (-2)$ (as is easily verified). In what follows, formulae are obtained for generalized inverses of $\mathbf{R}+\mathbf{STU}$ and for $\text{rank}(\mathbf{R}+\mathbf{STU})$ that apply without restriction.

Meyer ([10], Theorem 3.1) gave a formula for a generalized inverse of a partitioned matrix that applies without restriction. Let $\mathbf{G} = \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{pmatrix}$

represent the generalized inverse of the matrix **C** (introduced in Section 2) obtained by applying Meyer's formula. Further, letting $\mathbf{Q} = \mathbf{T} + \mathbf{TUR}^-\mathbf{ST}$ (as in Section 3), define $\mathbf{E_R} = \mathbf{I} - \mathbf{RR}^-$, $\mathbf{F_R} = \mathbf{I} - \mathbf{R}^-\mathbf{R}$, $\mathbf{X} = \mathbf{E_R ST}$, $\mathbf{Y} = \mathbf{TUF_R}$, $\mathbf{E_Y} = \mathbf{I} - \mathbf{YY}^-$, $\mathbf{F_X} = \mathbf{I} - \mathbf{X}^-\mathbf{X}$, $\mathbf{Z} = \mathbf{E_Y QF_X}$, and $\mathbf{Q}^* = \mathbf{F_X Z}^-\mathbf{E_Y}$. Then, assuming that (in applying Meyer's formula) the generalized inverse of $-\mathbf{X}$ is set equal to $-\mathbf{X}^-$, $\mathbf{G}_{11}$ equals

$$\mathbf{R}^- - \mathbf{R}^-\mathbf{STQ}^*\mathbf{TUR}^- - \mathbf{R}^-\mathbf{ST}(\mathbf{I} - \mathbf{Q}^*\mathbf{Q})\mathbf{X}^-\mathbf{E_R} - \mathbf{F_R}\mathbf{Y}^-(\mathbf{I} - \mathbf{QQ}^*)\mathbf{TUR}^- + \mathbf{F_R}\mathbf{Y}^-(\mathbf{I} - \mathbf{QQ}^*)\mathbf{QX}^-\mathbf{E_R} \tag{4.1}$$

And, in light of Theorem 2.5, we have the following theorem.

Theorem 4.1. Matrix (4.1) is a generalized inverse of $\mathbf{R} + \mathbf{STU}$.

Note that, like formula (3.1), formula (4.1) calls for the "inversion" of an $n \times q$ matrix and an $m \times p$ matrix [**R** and **Z**, in the case of formula (4.1); **R** and **Q**, in the case of formula (3.1)]. And, in addition, formula (4.1) calls for the "inversion" of the $n \times p$ matrix **X** and the $m \times q$ matrix **Y**. Note also that if $\Re(\mathbf{TU}) \subset \Re(\mathbf{R})$ and $\mathcal{C}(\mathbf{ST}) \subset \mathcal{C}(\mathbf{R})$, then $\mathbf{X} = \mathbf{0}$ and $\mathbf{Y} = \mathbf{0}$ (so that $\mathbf{F_X} = \mathbf{I}$ and $\mathbf{E_Y} = \mathbf{I}$ and consequently $\mathbf{Q}^*$ is an arbitrary generalized inverse of **Q**), and formula (3.1) can be obtained as a special case of formula (4.1) by setting $\mathbf{X}^- = \mathbf{0}$ and $\mathbf{Y}^- = \mathbf{0}$. Further, the formula given by Meyer ([9], Theorem 7) for a generalized inverse of $\mathbf{R} + \mathbf{STU}$, which is for the special case where **S** is a column vector, **U** is a row vector, and $\mathbf{T} = (1)$, can be obtained from formula (4.1).

If $\mathbf{R}^-$, $\mathbf{X}^-$, $\mathbf{Y}^-$, and $\mathbf{Z}^-$ are taken to be reflexive generalized inverses (of **R, X, Y,** and **Z**, respectively), then matrix (4.1) is a reflexive generalized inverse of $\mathbf{R} + \mathbf{STU}$, as can be shown by applying result (3.26) of Hartwig [5] to the matrix **C**.

Now, consider the rank of $\mathbf{R} + \mathbf{STU}$. Marsaglia and Styan ([4], formulae (8.6) and (8.7)) give two alternative formulae for the rank of a partitioned matrix, the first of which is also given by Meyer ([10], Theorem 4.1). Upon applying their formulae (both of which apply without restriction) to the matrix **C** and substituting the resultant expressions into expression (2.3), we obtain the following theorem and corollary.

Theorem 4.2.

$$\begin{aligned} \text{rank}(\mathbf{R} + \mathbf{STU}) &= \text{rank}(\mathbf{R}) + \text{rank}(\mathbf{X}) + \text{rank}(\mathbf{Y}) + \text{rank}(\mathbf{Z}) - \text{rank}(\mathbf{T}) \\ \text{rank}(\mathbf{R} + \mathbf{STU}) &= \text{rank}(\mathbf{R}) - [\text{rank}(\mathbf{T}) - \text{rank}(\mathbf{Q})] + \text{rank}(\mathbf{A}) \\ &\quad + \text{rank}(\mathbf{B}) + \text{rank}[(\mathbf{I} - \mathbf{AA}^-)\mathbf{XQ}^-\mathbf{Y}(\mathbf{I} - \mathbf{B}^-\mathbf{B})] \end{aligned} \tag{4.2}$$

where $\mathbf{A} = \mathbf{X}(\mathbf{I} - \mathbf{Q}^-\mathbf{Q})$ and $\mathbf{B} = (\mathbf{I} - \mathbf{QQ}^-)\mathbf{Y}$

Corollary 4.3.

$$\text{rank}(\mathbf{R} + \mathbf{STU}) \geq \text{rank}(\mathbf{R}) - [\text{rank}(\mathbf{T}) - \text{rank}(\mathbf{Q})] \tag{4.3}$$

with equality holding if and only if $\mathbf{X}(\mathbf{I} - \mathbf{Q}^{-}\mathbf{Q}) = \mathbf{0}$, $(\mathbf{I} - \mathbf{QQ}^{-})\mathbf{Y} = \mathbf{0}$, and $\mathbf{XQ}^{-}\mathbf{Y} = \mathbf{0}$.

Note that, for expression (4.2) to reduce to expression (3.11) [and for inequality (4.3) to hold as an equality], it is sufficient that $\Re(\mathbf{STU}) \subset \Re(\mathbf{R})$ and $\mathcal{C}(\mathbf{STU}) \subset \mathcal{C}(\mathbf{R})$. More generally, expression (3.11) is a lower bound for rank $(\mathbf{R} + \mathbf{STU})$.

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